数理解析研究所講究録
1012巻 1997年 82-91

ON α-CONVEX FUNCTIONS OF ORDER β OF RUSCHEWEYH TYPE-II

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ABSTRACT. The opject of the present paper is to establish an interesting result for certain multivalent functions in the unit disc.

KEY WORDS- p-Valent, convolution, Ruscheweyh derivative.

AMS (1991) Subject Classification. 30C45.

1. Introduction

Let $A(p)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \quad (p \in \mathbb{N} = \{1,2,\ldots\}) \tag{1.1}$$

which are analytic in the unit disc $U = \{z: |z| < 1\}$. For functions $f_j(z)$ $(j = 1,2)$ defined by

$$f_j(z) = z^p + \sum_{k=p+1}^{\infty} a_{k,j} z^k, \tag{1.2}$$

we define the convolution $f_1 * f_2(z)$ of functions $f_1(z)$ and $f_2(z)$ by

$$f_1 * f_2(z) = z^p + \sum_{k=p+1}^{\infty} a_{k,1}a_{k,2}z^k. \tag{1.3}$$

With the convolution above, we define

$$D^{n+p-1}f(z) = \left[\frac{z^p}{(1-z)^{n+p}}\right] * f(z) \quad (f(z) \in A(p)), \tag{1.4}$$

where n is any integer greater than $-p$. We note that

$$D^{n+p-1}f(z) = \frac{z^p(z^{n-1}f(z))^{(n+p-1)}}{(n+p-1)!}. \tag{1.5}$$

The symbol D^{n+p-1} when $p = 1$ was introduced by Ruscheweyh [11], and the symbol D^{n+p-1} was introduced by Goel and Sohi

[3]. Therefore, one called the symbol $D^{n+p-1}f(z)$ the (n+p-1)-th order Ruscheweyh derivative of f(z). It follows from (1.5) that

$$z(D^{n+p-1}f(z))' = (n+p)D^{n+p}f(z) - n\, D^{n+p-1}f(z). \qquad (1.6)$$

In [3] Goel and Sohi have introduced the class

$$K_{n+p-1} = \left\{ f(z) \in A(p) : \mathrm{Re}\left\{ \frac{D^{n+p}f(z)}{D^{n+p-1}f(z)} \right\} > \frac{1}{2} \right\} \qquad (1.7)$$

for $n \in \mathbb{N}_o = \mathbb{N} \cup \{0\}$ and $p \in \mathbb{N}$ and proved the theorem:

$$K_{n+p} \subset K_{n+p-1}. \qquad (1.8)$$

In [10] Soni had the generalization of Singh and Singh [9]:

$$R(n+p) \subset R(n+p-1), \qquad (1.9)$$

where

$$R(n+p-1) = \left\{ f(z) \in A(p) : \mathrm{Re}\left\{ \frac{D^{n+p}f(z)}{D^{n+p-1}f(z)} \right\} > \frac{n+p-1}{n+p} \right\} \qquad (1.10)$$

where n is any integer greater than -p.

A function $f(z) \in A(p)$ is said to be in the class $R_\beta(n+p-1,\alpha)$ if it satisfies the condition

$$\mathrm{Re}\left\{(1-\alpha)\frac{D^{n+p}f(z)}{D^{n+p-1}f(z)} + \alpha\,\frac{D^{n+p+1}f(z)}{D^{n+p}f(z)}\right\} > \beta \tag{1.11}$$

for all $z \in U$, $\alpha \geq 0$, $\beta < 1$, $p \in \mathbb{N}$ and n is any integer greater than $-p$. The class $R_{\beta}(n+p-1,\alpha)$ was introduced by Chen and Lan [2]. Also the class $R_{\beta}(n+p-1,\alpha)$ ($\alpha \geq 0$, $0 \leq \beta \leq \frac{1}{2}$, $p \in \mathbb{N}$, $n \in \mathbb{N}_0$) was studied by Kumar and Reddy [6].

We note that when $p = 1$, the class $R_{\frac{1}{2}}(n,\alpha) = M\,R_n(\alpha)$ was studied by Al-Amiri [1]. Also when $p = 1$ and $0 \leq \beta \leq \frac{1}{2}$, the class $R_{\beta}(n,\alpha) = T_{n,\beta}(\alpha)$ was studied by Goel and Sohi [3].

2. Main Result

In order to prove our main result, we recall here the following lemmas:

<u>Lemma 1 (Chen and Lan [2])</u>

For $p \in \mathbb{N}$, n is any integer greater than $-p$ and α is real

$$\text{(i)}\quad \frac{1}{2} \leq \frac{\{[2\beta(n+p+1)-3\alpha]+\sqrt{[2\beta(n+p+1)-3\alpha]^2+8\alpha(n+p+1-\alpha)}\}}{4(n+p+1-\alpha)} < 1 \tag{2.1}$$

when $\frac{1}{2} \leq \beta < 1$ and $\alpha \neq n+p+1$

$$\text{(ii)} \quad \frac{1}{2} \leq \frac{1}{(3-2\beta)} < 1 \tag{2.2}$$

when $\frac{1}{2} \leq \beta < 1$.

Lemma 2 (Miller [7]; Miller and Mocanu [8])

Let $\varphi(u,v)$ be a complex-valued function,

$$\varphi: D \longrightarrow C, \ D \subset C \times C \quad (C \text{ is the complex plane}),$$

and let $u = u_1 + iu_2$, $v = v_1 + iv_2$. Suppose that the function $\varphi(u,v)$ satisfies the following conditions:

(i) $\varphi(u,v)$ is continuous in D;

(ii) $(1,0) \in D$ and $\mathrm{Re}\{\varphi(1,0)\} > 0$;

(iii) $\mathrm{Re}\{\varphi(iu_2,v_1)\} \leq 0$ for all $(iu_2,v_1) \in D$ such that

$$v_1 \leq -\frac{(1+u_2^2)}{2}.$$

Let $q(z) = 1 + q_1 z + q_2 z^2 + \ldots$ be regular in the unit disc U, such that $(q(z),zq'(z)) \in D$ for all $z \in U$. If

$$\mathrm{Re}\left\{\varphi(q(z),zq'(z))\right\} > 0 \qquad (z \in U),$$

then

$$\mathrm{Re}\left\{q(z)\right\} > 0 \qquad (z \in U).$$

Applying Lemma 2, we derive the following:

Theorem 1. Let the function f(z) defined by (1.1) be in the class $R_\beta(n+p-1,\alpha)$ ($n > -p$, α is real), then

$$\mathrm{Re}\left\{\frac{D^{n+p}f(z)}{D^{n+p-1}f(z)}\right\} > \gamma(\alpha,\beta,n,p) \qquad (z \in U), \tag{2.3}$$

where

$$\gamma(\alpha,\beta,n,p) = \frac{[2\beta(n+p+1)-3\alpha]+\sqrt{[2\beta(n+p+1)-3\alpha]^2+8\alpha(n+p+1-\alpha)}}{4(n+p+1-\alpha)}$$

$$\text{if } \tfrac{1}{2} \le \beta < 1 - \frac{\alpha}{2(n+p+1)} < 1 \text{ and } \alpha \ne n+p+1$$

and

$$\gamma(\alpha,\beta,n,p) = \frac{1}{(3-2\beta)} \quad \text{if } \tfrac{1}{2} \le \beta < 1 \text{ and } \alpha = n+p+1.$$

Therefore, f(z) is in the class $R_\beta(n+p-1,\gamma(\alpha,\beta,n,p))$.

Proof. Define the function q(z) by

$$\frac{D^{n+p}f(z)}{D^{n+p-1}f(z)} = \gamma + (1-\gamma)q(z), \tag{2.4}$$

where $\gamma = \gamma(\alpha,\beta,n,p)$. Then $q(z) = 1 + q_1 z + q_2 z^2 + \ldots$ is regular in the unit disc U. Making use of the logarithmic differentiations of both sides of (2.4), we obtain

$$\frac{D^{n+p+1}f(z)}{D^{n+p}f(z)} = \frac{1}{(n+p+1)}\left\{1+(n+p)[\gamma+(1-\gamma)q(z)] + \frac{(1-\gamma)zq'(z)}{[\gamma+(1-\gamma)q(z)]}\right\}. \tag{2.5}$$

It follows from (2.4) and (2.5) that

$$\mathrm{Re}\left\{(1-\alpha)\frac{D^{n+p}f(z)}{D^{n+p-1}f(z)} + \alpha\frac{D^{n+p+1}f(z)}{D^{n+p}f(z)} - \beta\right\}$$

$$= \mathrm{Re}\left\{(1-\alpha)[\gamma+(1-\gamma)q(z)] + \frac{\alpha}{(n+p+1)}\left[1+(n+p)[\gamma+(1-\gamma)q(z)] + \frac{(1-\gamma)zq'(z)}{[\gamma+(1-\gamma)q(z)]}\right] - \beta\right\}$$

$$= \mathrm{Re}\left\{\frac{\alpha}{(n+p+1)} - \beta + \frac{(n+p+1-\alpha)}{(n+p+1)}[\gamma+(1-\gamma)q(z)] + \frac{\alpha(1-\gamma)zq'(z)}{(n+p+1)[\gamma+(1-\gamma)q(z)]}\right\} > 0. \tag{2.6}$$

Letting $u = u_1 + iu_2$, $v = v_1 + iv_2$, and

$$\varphi(u,v) = \frac{\alpha}{(n+p+1)} - \beta + \frac{(n+p+1-\alpha)}{(n+p+1)}[\gamma+(1-\gamma)u] + \frac{\alpha(1-\gamma)v}{(n+p+1)[\gamma+(1-\gamma)u]}, \tag{2.7}$$

we see that

(i) $\varphi(u,v)$ is continuous in $D = \left(C - \left\{\frac{\gamma}{\gamma-1}\right\}\right) \times C$;

(ii) $(1,0) \in D$ and $\mathrm{Re}\{\varphi(1,0)\} = 1 - \beta > 0$;

(iii) for all $(iu_2, v_1) \in D$ such that $v_1 \le -\dfrac{(1+u_2^2)}{2}$,

$$\mathrm{Re}\{\varphi(iu_2,v_1)\} = \frac{\alpha}{(n+p+1)} - \beta + \left(\frac{n+p+1-\alpha}{n+p+1}\right)\gamma + \frac{\alpha\gamma(1-\gamma)v}{(n+p+1)[\gamma^2+(1-\gamma)^2u_2^2]}$$

$$\le \frac{\alpha}{(n+p+1)} - \beta + \left(\frac{n+p+1-\alpha}{n+p+1}\right)\gamma + \frac{\alpha\gamma(1-\gamma)(1+u_2^2)}{2(n+p+1)[\gamma^2+(1-\gamma)^2u_2^2]}$$

$$\le 0 \qquad (2.8)$$

because $\frac{1}{2} \le \beta < 1 - \dfrac{\alpha}{2(n+p+1)} < 1$, $\alpha \ne (n+p+1)$ and $\frac{1}{2} \le \gamma < 1$ (see (2.1)). This implies that the function $\varphi(u,v)$ satisfies the conditions in Lemma 2. Thus applying Lemma 2, we have

$$\mathrm{Re}\left\{\frac{D^{n+p}f(z)}{D^{n+p-1}f(z)}\right\} > \gamma = \gamma(\alpha,\beta,n,p) \quad (z \in U).$$

Similarly, the other case of Theorem 1 can be proved by using (2.2). Hence the proof is completed.

Making $\alpha = 1$ in Theorem 1, we get

<u>Corollary 1</u>. If $f(z) \in R_\beta(n+p-1,1)$ $(n > -p)$, with $\frac{1}{2} \le \beta < 1 - \dfrac{1}{2(n+p+1)} < 1$, then

$$\mathrm{Re}\left\{\frac{D^{n+p}f(z)}{D^{n+p-1}f(z)}\right\} > \frac{[2\beta(n+p+1)-3]+\sqrt{[2\beta(n+p+1)-3]^2+8(n+p)}}{4(n+p)} \quad (z \in U) \qquad (2.9)$$

Letting $\beta = \beta' = \dfrac{3\alpha}{2(n+p+1)}$ in Theorem 1, we have

Corollary 2. If $f(z) \in R_{\beta'}(n+p-1,\alpha)$ $(n > -p)$, with $n+p+1 > 2\alpha$, then

$$\mathrm{Re}\left\{\frac{D^{n+p}f(z)}{D^{n+p-1}f(z)}\right\} > \sqrt{\frac{\alpha}{2(n+p+1-\alpha)}} \quad (z \in U). \tag{2.10}$$

Making $n = 1-p$ and $\alpha = 1$ in Theorem 1, we get

Corollary 3. If $f(z) \in R_{\beta}(0,1)$ with

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \frac{(4\beta-3) + \sqrt{(4\beta-3)^2 + 8}}{4} \quad (z \in U). \tag{2.11}$$

Remark. We note that Chen and Lan [2] have obtained the same results in Theorem 1 by applying Jack's Lemma [5].

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